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THE METHOD OF BOUNDARY INTEGRAL EQUATIONS IN UNSTEADY BOUNDARY-VALUE PROBLEMS OF UNCOUPLED THERMOELASTICITY†

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A development of the method of boundary integral equations for solving unsteady boundary-value problems of uncoupled thermoelasticity is presented. In the case of plane deformation, an algorithm for the numerical implementation of the method is presented and the results of calculations of a thermally stressed plane with apertures of circular (the test problem) and arched forms are given for the case when there is a specified unsteady heat flux on the boundary. © 2000 Elsevier Science Ltd. All rights reserved.

Unlike the case of dynamic problems in the theory of elasticity, there have only been a few studies of the dynamics of thermoelastic media with stress concentrators in the form of cavities and inclusions of various shapes. The class of particular solutions of the equations of thermoelasticity has been thoroughly investigated [1], mainly in the case of domains with a canonical form of the boundaries, where the methods of separation of variables and integral transforms are successfully used. The use of these methods in domains with a complex geometry is extremely restricted and frequently impossible. One of the potentially useful methods for solving such problems is the method of boundary integral equations, which was developed in [2–5] to solve static and quasistatic problems in thermoelasticity. There are only a few investigations of dynamic problems which take account of the inertial terms in the equations of motion which are based on this method and these are mainly of a theoretical nature. The present paper is concerned with the development of these problems.

1. FORMULATION OF THE PROBLEM

We consider a thermoelastic medium S^- which is bounded by a closed smooth Lyapunov surface S and $\mathbf{n}$ is the unit vector of the outward normal to S. The model of uncoupled thermoelasticity [1]

$$(c_1^2-c_2^2)\ u_{i,ij}+c_2^2\Delta u_j+F_j=u_{j,tt} \tag{1.1}$$

$$\Delta\theta(\mathbf{x},t)-k^{-1}\dot{\theta}(\mathbf{x},t)+g(\mathbf{x},t)=0 \tag{1.2}$$

$$\sigma_{ij}(\mathbf{x},t)=\mu(u_{i,j}+u_{j,i})+(\lambda u_{k,k}-\gamma\theta)\sigma_{ij} \tag{1.3}$$

$$F_j=G_j-\gamma\theta_{,j},\quad i,j,k=1,2;\quad \gamma=\alpha(3\lambda+2\mu)/\rho$$

is used to describe the motion of this medium.

Here, λ and μ are Lamé constants, ρ is the density, c_1, c_2 are the velocities of propagation of the longitudinal and transverse waves, α is the coefficient of linear thermal expansion, $k=\varkappa/(\rho c)$ is the thermal conductivity, c is the specific heat capacity, θ is the relative change in the absolute temperature, σ_{ij}, u_i are the components of the stress and strain tensors, which are connected by the Duhamel–Neumann relations (1.4), G_j are the components of the bulk force, and δ_{ij} is the Kronecker delta. Henceforth, derivatives with respect to the corresponding coordinates: $u_{ikj}=\partial^2u_i/\partial x_k\partial x_j$, $\theta_j=\partial\theta/\partial x_j$, are denoted by the symbol after the comma in the subscript, and time derivatives are denoted by a dot. Summation from 1 to 2 is carried out everywhere over repeated indices. At the initial instant of time $t=0$

$$u(\mathbf{x},0)=0,\quad \mathbf{x}\in S^-+S;\quad \dot{u}(\mathbf{x},0)=0,\quad \mathbf{x}\in S \tag{1.4}$$

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$$\theta(\mathbf{x},0)=0, \quad \mathbf{x}\in S^-+S \tag{1.5}$$

The acting loads and the heat flux on the boundary of the domain are known

$$\sigma_{ij}(\mathbf{x},t)n_j(\mathbf{x})=p_i(\mathbf{x},t), \quad \theta_{,j}n_j=q(\mathbf{x},t), \quad \mathbf{x}\in S \tag{1.6}$$

By virtue of the hyperbolic nature of system (1.1), the following conditions for the discontinuities in the derivatives in the wave fronts [6]

$$[\dot{u}_i+\upsilon\nu_j u_{i,j}]_F=0, \quad [\sigma_{ij}\nu_j+\rho\upsilon\dot{u}_i]_F=0 \tag{1.7}$$

must be satisfied for the class of solutions which are continuous with respect to the derivatives. Here $\upsilon = c_1, c_2$ is the rate of propagation of the discontinuity surface and ν are the direction cosines of the normal to it.

It is required to find σ_{ij}, u_i, θ in the medium for the specified boundary and initial conditions (1.4)–(1.6) and conditions (1.7).

2. FORMULATION OF THE PROBLEM IN LAPLACE TRANSFORM SPACE

A Laplace transform with respect to t

$$\bar{u}_i(\mathbf{x},p)=\int_0^{+\infty}u_i(\mathbf{x},t)e^{-pt}dt, \quad \operatorname{Re}p\geqslant p_0>0$$

is used to solve the problem. The problem can then be subdivided into two boundary-value problems. We first solve the following problem.

The boundary-value problem for determining the temperature field. Equation (1.2) is transformed to the form

$$\Delta\bar{\theta}(\mathbf{x},p)-k^{-1}p\bar{\theta}(\mathbf{x},p)+\bar{g}(\mathbf{x},p)=0 \tag{2.1}$$

Initial condition (1.5) is transformed into the asymptotic condition

$$\lim_{p\to\infty}p\bar{\theta}(\mathbf{x},p)=0, \quad \mathbf{x}\in S^-+S$$

The boundary condition has an analogous form

$$\frac{\partial\bar{\theta}}{\partial\mathbf{n}}=\bar{q}(\mathbf{x},p), \quad \mathbf{x}\in S \tag{2.2}$$

After the temperature field has been determined, we solve the following problem.

The boundary-value problem for determining the thermoelastic displacements $\bar{u}_e$. The equations of motion

$$(c_1^2-c_2^2)\bar{u}_{i,ij}+c_2^2\Delta\bar{u}_j-\gamma\bar{\theta}_{,j}+\bar{G}_j=p^2\bar{u}_j, \quad i,j=1,2 \tag{2.3}$$

contain the temperature gradient, which is now known.

Initial conditions (1.4) are transformed into the asymptotic conditions

$$\lim_{p\to\infty}p\bar{u}(\mathbf{x},p)=0, \quad \mathbf{x}\in S^-+S; \quad \lim_{p\to\infty}p^2\bar{u}(\mathbf{x},p)=0, \quad \mathbf{x}\in S^-$$

Boundary conditions (1.6) have the form

$$\bar{\sigma}_{ij}(\mathbf{x},p)n_j(\mathbf{x})=\bar{p}_i(\mathbf{x},p), \quad \mathbf{x}\in S$$

The method of boundary integral equations (BIEs) is used to solve both problems.

3. BOUNDARY INTEGRAL EQUATIONS FOR A TIME-DEPENDENT PROBLEM OF UNCOUPLED THERMOELASTICITY

The methods of the theory of generalized functions is used to determine the temperature field using the technique which has previously been described in [6]. In the space of generalized functions, the equation of the temperature field (2.1) has the form

$$\Delta\hat{\theta}(\mathbf{x},p)-pk^{-1}\hat{\theta}(\mathbf{x},p)=-n_j(\mathbf{x})\partial_j\overline{\theta}(\mathbf{x},p)\delta_S(\mathbf{x})-$$
$$-\partial_j(n_j(\mathbf{x})\overline{\theta}(\mathbf{x},p)\delta_S(\mathbf{x}))-\overline{g}(\mathbf{x},p)H_S^-(\mathbf{x}) \tag{3.1}$$
$$\hat{\theta}(\mathbf{x},p)=\overline{\theta}(\mathbf{x},p)H_S^-(\mathbf{x}),\quad H_S^-(\mathbf{x})=\begin{cases}1, & \mathbf{x}\in S^-\\ 1/2, & \mathbf{x}\in S\\ 0, & \mathbf{x}\in S^+\end{cases}$$

where $f(\mathbf{x})\delta_S(\mathbf{x})$ is a singular generalized function—a single layer in the set S, with density $f(\mathbf{x})$ [7] and $H_s(\mathbf{x})$ is the characteristic function of the set $\bar{S}^-$.

By convolution of the right-hand side of (3.1) with Green's function $\theta^*(\mathbf{x},p)$, we obtain an analogue of Green's formula in the space of generalized functions

$$\hat{\theta}(\mathbf{x},p)=\overline{\theta}^*(x,p)*\overline{q}(\mathbf{x},p)\delta_S(\mathbf{x})+\partial_j\overline{\theta}^*(\mathbf{x},p)*n_j(\mathbf{x})\overline{\theta}(\mathbf{x},p)\delta_S(\mathbf{x})+$$
$$+\overline{\theta}^*(\mathbf{x},p)*\overline{g}(\mathbf{x},p)H_S^-(\mathbf{x}) \tag{3.2}$$

where

$$\overline{\theta}^*(\mathbf{x},p)=(2\pi)^{-1}K_0(\|\mathbf{x}\|\sqrt{p/k}) \tag{3.3}$$

We write the integral representation of the transform of the temperature field (3.2), taking account of (3.3), in the form

$$2\pi\overline{\theta}(\mathbf{x},p)H_S^-(\mathbf{x})=\int_{S^-}K_0(r\sqrt{p/k})\overline{g}(\mathbf{y},p)dv(\mathbf{y})+\int_S K_0(r\sqrt{p/k})\overline{q}(\mathbf{y},p)ds(\mathbf{y})-$$
$$-\sqrt{p/k}\int_S K_1(r\sqrt{p/k})r_{,j}n_j(\mathbf{y})\overline{\theta}(\mathbf{y},p)ds,\quad r=\|\mathbf{x}-\mathbf{y}\| \tag{3.4}$$

where $K_n(pr/c_2)$ are McDonald functions. For $\mathbf{x}\in S$, the third integral on the right-hand side is taken in the sense of the principal value.

Relation (3.4) for $\mathbf{x}\in S$ is a singular BIE which enables us to find the transform of the temperature $\overline{\theta}(\mathbf{x},p)$ in S if the heat flux $\overline{q}(\mathbf{x},p)$ is known. Numerical methods are used to solve these equations in the case of an arbitrary contour S. After solving the BIE using Eq. (3.4), $\overline{\theta}(\mathbf{x},p)$ is recovered for $\mathbf{x}\in S^-$.

4. DETERMINATION OF THE STRAIN AND STRESS FIELDS

If the temperature field $\overline{\theta}(\mathbf{x},p)$ is known, one can solve the second boundary-value problem, since the temperature field now appears in (2.3) in the form of a known "mass force". We next assume that $\overline{G}_j=0$. The required solution can be represented in the form

$$\overline{u}(\mathbf{x},p)=\overline{u}^0(\mathbf{x},p)+\overline{u}^r(\mathbf{x},p),\quad \overline{\sigma}_{ij}(\mathbf{x},p)=\overline{\sigma}_{ij}^0+\overline{\sigma}_{ij}^r \tag{4.1}$$

where $\bar{u}^0(\mathbf{x},p)$ is the solution of the homogeneous equation (we shall call it the "elastic solution" which corresponds to (1.3), $F_j^-=0$) and $\bar{u}^r(\mathbf{x},p)$ is a particular solution of the inhomogeneous equation (2.3) (which is conventionally called the "temperature solution"). The potential method, which we previously developed to solve problems in elastodynamics [6] is used here to construct the BIE for determining the displacements $\bar{u}^0(\mathbf{x},p)$.

The function $\bar{u}^0(\mathbf{x},p)$ is sought in the form of the potential of the single layer

$$\bar{u}_i^0(\mathbf{x},p)=\int_S \bar{U}_{ij}(\mathbf{x}-\mathbf{y},p)\bar{\varphi}_j(\mathbf{y},p)dS(\mathbf{y})$$

where $\bar{U}_{ij}(\mathbf{x},p)$ is Green's tensor of the equations of elastodynamics [6, 8] and the density $\bar{\varphi}_j(\mathbf{y},p)$ is the solution of a BIE of the form

$$0.5\bar{\varphi}_i(\mathbf{x},p)=\int_S \bar{\Gamma}_{ik}(\mathbf{x},\mathbf{y},p)\bar{\varphi}_k(\mathbf{y},p)dS(\mathbf{y})=\bar{f}_i(\mathbf{x},p),\quad \mathbf{x}\in S$$

Here,

$$\bar{f}_i(\mathbf{x},p)=\bar{p}_i(\mathbf{x},p)-\bar{\sigma}_{ij}^r(\mathbf{x},p)n_j(\mathbf{x}),\quad \mathbf{x}\in S \tag{4.2}$$

The particular solution of Eqs (2.3) has the form of a convolution of the "mass force" generated by the temperature field with Green's tensor $\hat{U}_{ij}(\mathbf{x},p)$ which (when $\hat{G}_j=0$) reduces to the form

$$\bar{u}_i^r(\mathbf{x},p)=\gamma p(2\pi c_1^3)^{-1}\int_{S^-}\bar{\theta}(\mathbf{y},p)K_1(pr/c_1)r_{,i}dv(\mathbf{y}) \tag{4.3}$$

The thermal stresses generated by these displacements are given by the formula

$$\begin{aligned}&\bar{\sigma}_{ij}^r(\mathbf{x},p)=\bar{S}_{ij}^k(\mathbf{x},p)*\bar{G}_k(\mathbf{x},p)-\gamma\bar{\theta}(\mathbf{x},p)H_S^-(\mathbf{x})\delta_{ij}-\\&-\gamma\bar{S}_{ij}^k(\mathbf{x},p)*\partial_k\bar{\theta}(\mathbf{x},p)H_S^-(\mathbf{x})+\gamma\bar{S}_{ij}^k(\mathbf{x},p)*\bar{\theta}(\mathbf{x},p)n_k\delta_S(\mathbf{x})\\&\bar{S}_{ij}^k(\mathbf{x},p)=\lambda\bar{U}_{m,m}^k\delta_{ij}+\mu(\bar{U}_{i,j}^k+\bar{U}_{j,i}^k)\end{aligned} \tag{4.4}$$

Here $\bar{S}_{ij}^k(\mathbf{x},p)$ is the fundamental stress tensor generated by $\hat{U}_{ij}(\mathbf{x},p)$.

We write relation (4.4) in the integral form

$$\begin{aligned}&\bar{\sigma}_{ij}^r(\mathbf{x},p)=-\gamma\bar{\theta}(\mathbf{x},p)H_S^-(\mathbf{x})\sigma_{ij}-\gamma\int_{S^-}\bar{S}_{ij}^k(\mathbf{x},\mathbf{y},p)\frac{\partial\bar{\theta}(\mathbf{y},p)}{\partial y_k}dv(\mathbf{y})\\&+\gamma\int_S\bar{S}_{ij}^k(\mathbf{x},\mathbf{y},p)\bar{\theta}(\mathbf{y},p)n_k(\mathbf{y})ds(\mathbf{y})\end{aligned} \tag{4.5}$$

For $\mathbf{x}\in S^-$, all the integrals exist but, when $\mathbf{x}\in S$, the last integral in (4.5) has a strong singularity since

$$K_1(r\sqrt{p/k})\sim(r\sqrt{p/k})^{-1}\text{ when }r\to 0$$

Formula (4.5) contains temperature derivatives and is inconvenient for numerical calculations. It is transformed to a form which does not contain $\bar{\theta}_{,k}$.

We now consider relation (4.5) for the case when $\mathbf{x}\in S^-$. Using Gauss' theorem and regularizing the volume integrals which arise here, we obtain a formula for calculating the thermal stresses when $\mathbf{x}\in S^-$

$$\begin{aligned}&\bar{\sigma}_{ij}^r(\mathbf{x},p)=\gamma\Big[\int_S[\bar{\theta}(\mathbf{y},0)-\bar{\theta}(\mathbf{x},p)]\bar{\Phi}_{ij}^1(\mathbf{x}-\mathbf{y},p)dv(\mathbf{y})+\\&+\bar{\theta}(\mathbf{x},p)\int_S\bar{S}_{ij}^k(\mathbf{x},\mathbf{y},p)n_k(\mathbf{y})ds(\mathbf{y})\Big]-\gamma\bar{\theta}(\mathbf{x},p)H_S^-(\mathbf{x})\delta_{ij}\\&\bar{\Phi}_{ij}^1(\mathbf{x},\mathbf{y},p)=\frac{\partial\bar{S}_{ij}^k}{\partial y_k}=\frac{\rho p^2}{2\pi c_1^2}\left(\left((2c^2-1)K_0\left(\frac{pr}{c_1}\right)+\right.\right.\\&\left.\left.+2c^2\frac{c_1}{pr}K_1\left(\frac{pr}{c_1}\right)\right)\delta_{ij}-2c^2\left(K_0\left(\frac{pr}{c_1}\right)+2\frac{c_1}{pr}K_1\left(\frac{pr}{c_1}\right)\right)r_{,i}r_{,j}\right)\end{aligned} \tag{4.6}$$

Relation (4.6) does not contain temperature derivatives and is more convenient for numerical calculations.

Formulae (4.6) cannot be used when $\mathbf{x} \in S$ since the second integral on the right does not exist. Taking the limit in (4.6) with respect to $\mathbf{x} \to \mathbf{x}^*$ ($\mathbf{x} \in S^-$, $\mathbf{x}^* \in S$) and using a previously obtained formula [6, p. 144] we obtain $\bar{\sigma}^r_{ij}(\mathbf{x}, p)$ on the boundary.

As a result, we have†

$$\bar{\sigma}^r_{ij}(\mathbf{x},p) = \gamma\Bigg[\int_{S^-}[\bar{\theta}(\mathbf{y},p) - \bar{\theta}(\mathbf{x},p)]\bar{\Phi}^1_{ij}(\mathbf{x},\mathbf{y},p)dv(\mathbf{y}) + \tag{4.7}$$

$$+V.p.\int_S \bar{S}^k_{ij}(\mathbf{x},\mathbf{y},p)\bar{\theta}(\mathbf{x},p)n_k(\mathbf{y})ds(\mathbf{y}) + c^2\bar{\theta}(\mathbf{x},p)n_i n_j - \bar{\theta}(\mathbf{x},p)(1/2 + c^2\delta_{ij})\Bigg],$$

for the thermal stresses when $\mathbf{x} \in S$.

Formula (4.7) enables us to find the right-hand side of BIE (4.2). After solving these equations, we determine the strains and stresses in the medium using relations (4.1).

5. AN ALGORITHM FOR THE NUMERICAL SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF THERMOELASTICITY

The proposed algorithm consists of severe steps: interpolation of the boundary contour using cubic splines, construction of the discrete analogues of the BIEs by a piecewise-constant approximation of the temperature by means of a linear approximation $\varphi(\mathbf{x}, p)$ in each boundary element, the use of Gauss' quadrature formulae to evaluate the surface and volume integrals, the solution of the discrete analogues of the BIEs for the temperature for a specified sequence of values of the Laplace transform parameter $\{p_k\}$, calculation of the transform of the "temperature" strains and stresses, solution of the discrete analogue of the BIE for the density $\varphi(\mathbf{x}, p)$ for the specified sequence $\{p_k\}$ and calculation of the originals of the resulting strains and stresses by numerical inversion of the Laplace transform.

Discrete analogues of the BIEs were obtained in the form of a system of linear algebraic equations, the order of which is determined by the number of boundary elements N (see the paper cited in the footnote).

Evaluation of the volume integrals (4.3) and (4.7) presents a certain amount of difficulty. The calculations were carried out for a plane with an aperture of arbitrary form (external problems). In order to do this, the domain S^- was subdivided into zones which were commensurate, close to the boundary, with the size of the boundary elements. Cubic functions in the form of a plane element were used when integrating inside the curvilinear elements.

Two inversion schemes were used for the numerical inversion of the transforms of the solutions: the Papoulis scheme [9], for which a knowledge of the transform of the solution on the real axis is required, and a discrete Laplace transform [10] for complex values of p.

An analytical solution of the problem of the thermally stressed state of a plane weakened by a circular aperture (see the paper cited in the footnote), with the boundary conditions when $R = \|\mathbf{x}\| = 1$

$$q(\mathbf{x},t) = H(t), \quad \sigma_{ij}(\mathbf{x},t)n_j(\mathbf{x}) = 0, \quad i,j = 1,2 \tag{5.1}$$

was used to test the algorithm. In (5.1), ($H(t)$ is the Heaviside function). The following values of the dimensionless parameters were used in the calculations: $\nu = 0.25$, $\rho = 1$, $c_1 = 1$, $\gamma = 1$, $k = 1$.

Calculations of the temperature transform $\theta(\mathbf{x}, p)$ showed that the relative error ε decreases as the number of boundary elements N is increased. For instance, at $p = 0.1$, $R = 1$ an increase in N from 12 to 60 leads to a decrease in ε from 0.015 to 0.003, at $p = 0.1$, $R \geqslant 1.5$ it leads to a decrease in ε from 0.0014 to 0.0002, at $p = 2$, $R = 1$ to a decrease in ε from 0.057 to 0.011, and at $p = 2$, $R \geqslant 1.5$ to a decrease in ε from 0.017 to 0.003. The same is observed in the case of the value of $\sigma_{\theta\theta}$ calculated at $R = 1$ (on the boundary of the contour).

The values of the temperature and the shear stresses, calculated using the Papoulis scheme and using the discrete Laplace transform were practically identical: the relative error was less than 0.01.

In order to illustrate the possibilities of the algorithm developed above, problems were solved on the determination of the thermally stressed state of a plane, weakened by an arch-shaped aperture (Fig. 1) acted upon by a constant heat flux of the form (5.1) (see the paper cited in the footnote) and a pulsed heat flux

†DADAYEVA, A. N., The method of boundary integral equations in a time-dependent boundary-value problem of thermoelasticity. Candidate dissertation, Almaty 1995.

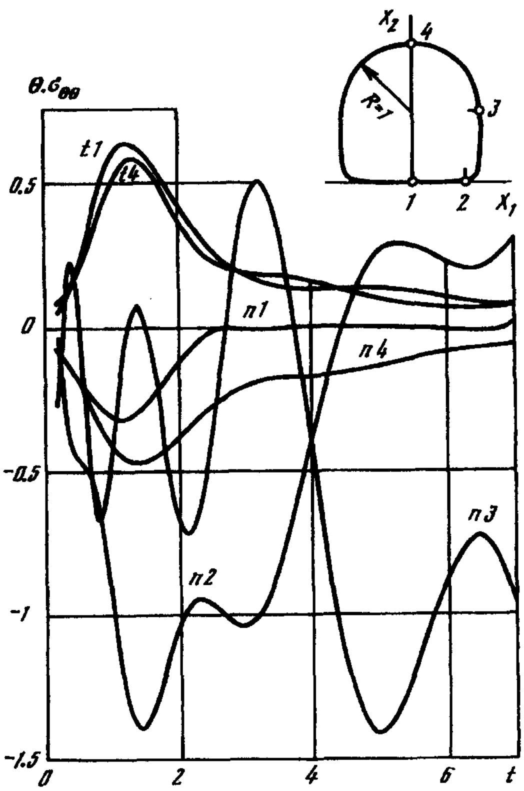

Fig. 1.

$$q(\mathbf{x},t) = tH(t)H(1-t) - (t-2)H(t-1)H(2-t) \tag{5.2}$$

Here, the transform was inverted using the Papoulis scheme.

The results of the calculations at the characteristic points of the arch in the case of (5.2) are shown in Fig. 1. Curves $t1$–$t4$ correspond to the values of the temperature $\theta(\mathbf{x}, t)$ at the four points of the arch: (0, 0), (0, 9, 0), (1, 1), (0, 2) (curves $t2$ and $t3$ pass between $t1$ and $t4$ and are not shown in Fig. 1) and $n1$–$n4$ are the values of the normal stresses in the tangential planes of these points. The temperature and stresses at the points of the arch where the curvature is continuous are of a pulsed nature, similar to (5.2). An oscillatory process occurs at points 2 and 3, where the curvature has a discontinuity. Note that instability in the numerical recovery of the originals occurs at long times, which does not enable the asymptotic behaviour of the solutions with time to be followed. However, at short times, which are characteristic of transients, the algorithm for a calculation based on the BIE method is quite stable.

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